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Inter-parametric Tolerance Analysis of a Multiobjective Linear Programming

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In this paper, global sensitivity analysis of a Multiple Objective Linear Programming problem is performed. The analysis takes the views from the utilities of management and control and is focused on the global perturbations on the weights, the cost coefficients, and the right-hand sides. Different coefficients are determined by different factors (parameters). Because it is different from the multi-parametric programming, we call it the inter-parametric tolerance analysis to emphasize the interactions of the parameters. Besides, a geometric approach is adopted to find the critical region. As regards the effective control and management, a bi-level criterion based on two respective factors stated below is proposed to determine comparatively the sensitive parameters: The first level is taken on the smaller relative tolerance-ratio; if they are equal, the second level of analysis is based on the larger unit-contribution to the objectives. This is done by the direct measurement as well as from the concept of the marginal rate of substitution. Theoretical development is incorporated with numerical illustrations.

1. INTRODUCTION

Consider a Multiple Objective Linear Program (MOLP) in general form as

$$\begin{aligned} \operatorname{Max} Z = [\mathbf{z}^1, ..., \mathbf{z}^k, ..., \mathbf{z}^K] &= [\mathbf{c}^1\mathbf{x}, ..., \mathbf{c}^k\mathbf{x}, ..., \mathbf{c}^K\mathbf{x}] \\ \text{s.t.} \quad A\mathbf{x} &\leqslant \mathbf{b} \\ \mathbf{x} &\geqslant \mathbf{0}, \end{aligned} \tag{1}$$

where $A = [a_{ij}]$, $i = 1, ..., m$; $j = 1, ..., n$ is an $m \times n$ matrix; $\mathbf{b} = [b_i]$, $i = 1, ..., m$, and $\mathbf{c}^k = [c_j^k]$, $k = 1, ..., K$, $j = 1, ..., n$, are the column and row vectors, respectively. When the levels of importance on the criteria are provided by a decision maker (DM), the values of the weights can be specified by the nonnegative values denoted by λ^k, $k = 1, ..., K$, $\sum \lambda^k = 1$,

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and model (1) will be transformed into a single weighted objective linear program (LP). Then, one can use the simplex method [5] to solve this problem.

However, a DM usually is interested not only in the optimal solution for the given values of each λ^k, c_j^k, and b_i, but also in what may happen to the optimal solution when some of these values are changed. Then, this will rely on a sensitivity analysis of the weights, the cost coefficients, or the right-hand sides (RHSs). When one focuses on the largest allowable intervals of the impact factors (parameter) considered in any given value for the current optimal solution, this is an approach of tolerance analysis. Therefore, if there is more than one parameter considered in any given value of λ, c, and b, it is a multi-parametric tolerance analysis [13–16]; otherwise, it is a single-parametric tolerance analysis. Conventionally, if more than one given value is analyzed simultaneously, then these impact factors are taken to be the same for simplicity. This is different from parametric programming in that the latter is used to analyze the trend of the optimal solutions with respect to the changes of any λ, c, b.

Both approaches have been studied by many researchers. However, most of the studies on tolerance analysis are focused on a single objective model. For instance, Gal [11] has developed a series of single and multiple parametric analyses in an LP to find the largest allowable regions (the critical regions) for the individual cost coefficients, the constraint-matrix, and the RHSs. Gass and Saaty [8], Wendell [13–15], Hansen, Labbe, and Wendell [16], and Wang and Huang [12] then considered multi-parametric analysis on a single objective linear programming to find a maximal tolerance region with different criteria.

Parametric programming with an MOLP was recently considered. Most of the existing methods, such as Deshpande [1], were developed for an individual coefficient and solved by the revised simplex method.

Because the determination of the resources b and the cost coefficients c will jointly affect a DM's preference structure represented by the weights λ and all of them, in turn, will affect the final solution, studies of the sensitivities of these given values should look at their global behavior. Besides, using simplex tableau can not cope with the global changes of the coefficients with non-linear forms, therefore an alternative approach to analyzing its geometric properties is proposed so that more intuitive insight of the problem can be obtained. Furthermore, different coefficients are normally affected by different factors, therefore, different parameters will be defined respectively. For simplicity, in this study, a single parameter will be considered for each type of coefficients and their inter-relations will particularly be studied. Thus, we call it "Inter-parametric Tolerance Analysis."

Finally, it is noted that a basic objective of sensitivity analysis is to identify the sensitive parameters so that special care can be taken to manipulate them more effectively. To detect the sensitive parameters, a bi-level criterion is based: (1) the smaller the relative tolerance-ratio, the more sensitive is the degree of a parameter; if two parameters are equally sensitive, for the sake of optimality, (2) the larger the corresponding unit-contribution to the objective functions, the more care should be taken with the parameter. Then, a multi-stage analysis corresponding to these criteria is performed from both direct measurement and their Marginal Rate of Substitution (MRS).

2. Theoretical Development

To proceed with sensitivity analysis of the weights, the cost coefficients, and the RHSs in problem (1), we first focus on the following problem where different single-parameters are defined for different coefficients,

$$\begin{aligned} \text{Max } Z &= \sum_{k=1}^{K} (\lambda^k + \mu\rho^k)(\mathbf{c}^k + \theta\boldsymbol{\alpha}^k)\,\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \leqslant \mathbf{b} + \delta\boldsymbol{\beta} \\ & \mathbf{x} \geqslant \mathbf{0}, \end{aligned} \tag{2}$$

where $\boldsymbol{\alpha}^k = [\alpha_j^k]$, $\boldsymbol{\beta} = [\beta_i]$, and $\boldsymbol{\rho} = [\rho^k]$, $\Sigma\rho^k = 0$, are the given vectors with respect to the parameters θ, δ, and μ.

Here, we can interpret μ, θ, and δ as the allowed perturbations that we would like to estimate for given vectors $\boldsymbol{\rho}$, $\boldsymbol{\alpha}^k$, and $\boldsymbol{\beta}$. When $\rho^k = \lambda^k$ for $k = 1, ..., K$, $\alpha_j^k = c_j^k$ for $j = 1, ..., n$, $k = 1, ..., K$, and $\beta_i = b_i$ for $i = 1, ..., m$, we can interpret μ, θ, and δ as percentage perturbations from the estimated values of each λ^k, c_j^k, and b_i, respectively. When any one of ρ^k, α_j^k, and β_i equals 1, then μ, θ, and δ can represent additive variations from λ^k, c_j^k, and b_i, respectively. Alternatively, if the relative importance of the different perturbations is desired, for instance, if $\beta_1 = b_1$ and $\beta_2 = 2b_2$, then the value of δ will correspond to twice as large of a percentage variation in b_2 with respect to b_1 [13–16].

2.1. *Determination of the Critical Region*

Before we proceed with the analysis of the simultaneous perturbation on the weights, the cost coefficients, and the RHSs, let us define some notations below. Note that once they are defined, they will be used throughout the paper until otherwise stated:

i, p: index of constraints from 1 to m;

j: index of decision variables from 1 to n;

X_B^*: the set of the optimal basic variables;

$$I' = \{i \mid x_i \in X_B^*\} \text{ and}$$

$$I = \{i \mid x_i \text{ is an original variable and } x_i \in X_B^*\};$$

b_i^*: the value of the RHSs, which corresponds to the row x_i of the final simplex tableau;

s_{ip}^*: the value that corresponds to the row $x_i \in X_B^*$ and the column $(n+p)$ of the final simplex tableau;

y_i^*: the shadow price that is the $(n+i)$th element on the reduced cost row of the final simplex tableau;

a_{ij}^*: the value corresponding to the row $x_i \in X_B^*$ and the column j of the final simplex tableau.

It is noted that the weights must be the real values varying from 0 to 1. Thus, we have the weighting constraint:

$$\operatorname{Max}\left\{\operatorname*{Max}_{k \in K^-}\left\{\frac{1-\lambda^k}{\rho^k}\right\}, \operatorname*{Max}_{k \in K^+}\left\{\frac{-\lambda^k}{\rho^k}\right\}\right\}$$
$$\leqslant \mu \leqslant \operatorname{Min}\left\{\operatorname*{Min}_{k \in K^+}\left\{\frac{1-\lambda^k}{\rho^k}\right\}, \operatorname*{Min}_{k \in K^-}\left\{\frac{-\lambda^k}{\rho^k}\right\}\right\}$$

where $K^- = \{k \mid \rho^k < 0\}$ and $K^+ = \{k \mid \rho^k > 0\}$.

Then from problem (2), we have

$$\Delta C_j = \sum_{k=1}^{K} (\lambda^k \alpha_j^k \theta + \rho^k \mu c_j^k + \rho^k \mu \alpha_j^k \theta), \qquad j = 1, \ldots, n$$

and

$$\Delta b_i = \beta_i \delta, \qquad i = 1, \ldots, m.$$

TABLE I

The Perturbed Optimal Tableau of Problem (2)

	x_r	x_{n+p}	RHS
Z	$Z_r^* - C_r + \Delta(Z_r^* - C_r)$ $+ \sum_{i \in I'} [\Delta(Z_i^* - C_i)]\, a_{ir}^*$	y_p^* $+ \sum_{i \in I'} [\Delta(Z_i^* - C_i)]\, s_{ip}^*$	$(Z^*)'$
x_i	a_{ir}^*	s_{ip}^*	$b_i^* + \Delta b_i^*$

Note: $(Z^*)' = Z^* + \Delta Z^* + \sum_{i \in I'} \{[\Delta(Z_i^* - C_i)](b_i^* + \Delta b_i^*)\}$.

By using the general formulations of the simplex method [5], we have

$$\Delta(Z_j^* - C_j) = -\sum_{k=1}^{K} (\lambda^k \alpha_j^k \theta + \rho^k \mu c_j^k + \rho^k \mu \alpha_j^k \theta), \qquad j = 1, ..., n;$$

$$\Delta Z^* = \sum_{i=1}^{m} \beta_i y_i^* \delta,$$

and

$$\Delta b_i^* = \sum_{p=1}^{m} \beta_p s_{ip}^* \delta, \qquad \forall i \in I'.$$

So, when the perturbation occurs, a new optimal tableau can be obtained by using the Gauss–Jordan method as shown in Table I.

Since it is still feasible and optimal, we have the relations

$$\begin{aligned} &\theta\mu\left[\sum_{i\in I}\left(\sum_{k=1}^{K}\rho^k\alpha_i^k\right)a_{ir}^* - \sum_{k=1}^{K}\rho^k\alpha_r^k\right] \\ &\quad + \theta\left[\sum_{i\in I}\left(\sum_{k=1}^{K}\lambda^k\alpha_i^k\right)a_{ir}^* - \sum_{k=1}^{K}\lambda^k\alpha_r^k\right] \\ &\quad + \mu\left[\sum_{i\in I}\left(\sum_{k=1}^{K}\rho^k c_i^k\right)a_{ir}^* - \sum_{k=1}^{K}\rho^k c_r^k\right] + Z_r^* - C_r \geqslant 0, \qquad \forall r \in R \quad (3) \end{aligned}$$

$$\begin{aligned} &\theta\mu\left[\sum_{i\in I}\left(\sum_{k=1}^{K}\rho^k\alpha_i^k\right)s_{ip}^*\right] + \theta\left[\sum_{i\in I}\left(\sum_{k=1}^{K}\lambda^k\alpha_i^k\right)s_{ip}^*\right] \\ &\quad + \mu\left[\sum_{i\in I}\left(\sum_{k=1}^{K}\rho^k c_i^k\right)s_{ip}^*\right] + y_p^* \geqslant 0, \qquad \forall p \in P \qquad (4) \end{aligned}$$

$$\delta\left(\sum_{p=1}^{m}\beta_p s_{ip}^*\right) + b_i^* \geqslant 0, \qquad \forall i \in I', \qquad (5)$$

where $R = \{r \mid x_r$ is an original variable and $x_r \notin X_B^*\}$ and $P = \{p \mid x_{n+p}$ is a slack variable and $x_{n+p} \notin X_B^*\}$.

Note that Eq. (5) is independent of Eqs. (3) and (4). So let us look at Eq. (5) first by the analysis

(a) if $I^+ = \left\{i \mid i \in I', \sum_{p=1}^{m} \beta_p s_{ip}^* > 0\right\}$, then

$$\delta \geqslant \frac{-b_i^*}{\sum_{p=1}^{m} \beta_p s_{ip}^*}, \qquad \forall i \in I^+; \qquad (6)$$

(b) if $I^= = \left\{i \mid i \in I', \sum_{p=1}^{m} \beta_p s_{ip}^* = 0\right\}$, then

$$\delta \in \mathbb{R}, \qquad \forall i \in I^=; \qquad (7)$$

(c) if $I^- = \left\{ i \mid i \in I', \sum_{p=1}^{m} \beta_p s_{ip}^* < 0 \right\}$, then

$$\delta \leqslant \frac{-b_i^*}{\sum_{p=1}^{m} \beta_p s_{ip}^*}, \qquad \forall i \in I^-. \tag{8}$$

Thus, from the above we have the general form of the largest tolerance interval for the parameter δ with the basis X_B^* as

$$\operatorname*{Max}_{i \in I^+} \left\{ \frac{-b_i^*}{\sum_{p=1}^{m} \beta_p s_{ip}^*} \right\} \leqslant \delta \leqslant \operatorname*{Min}_{i \in I^-} \left\{ \frac{-b_i^*}{\sum_{p=1}^{m} \beta_p s_{ip}^*} \right\}. \tag{9}$$

As regards parameters μ and θ in Eqs. (3) and (4), let us consider the general form of a second-degree equation

$$D\theta^2 + E\mu\theta + F\mu^2 + G\theta + H\mu + M = 0 \tag{10}$$

with its geometric properties as

$$\begin{aligned} &\text{ellipse} && \text{if } E^2 + 4DF < 0, \\ &\text{hyperbola} && \text{if } E^2 + 4DF > 0, \\ &\text{parabola} && \text{if } E^2 + 4DF = 0. \end{aligned} \tag{11}$$

Now, when the equalities of (3) and (4) hold, they are equivalent to $D=0$ and $F=0$ of Eq. (10), which is a hyperbola as

$$E\mu\theta + G\theta + H\mu + M = 0 \qquad \text{with} \quad \left(\theta + \frac{H}{E}\right)\left(\mu + \frac{G}{E}\right) = \frac{GH - EM}{E^2}. \tag{12}$$

If additionally $E=0$ is set, it is a parabola and may degenerate to a straight line.

The aforesaid hyperbola's asymptotes are $\theta = -H/E$ and $\mu = -G/E$. These two lines are mutually perpendicular and parallel to the coordinate lines. When rotating the axes through an angle $\pi/4$ and using the transformation

$$\theta = \frac{1}{\sqrt{2}}\theta' - \frac{1}{\sqrt{2}}\mu'$$

$$\mu = \frac{1}{\sqrt{2}}\theta' + \frac{1}{\sqrt{2}}\mu',$$

we have

$$\text{(A)} \quad \frac{(\theta' + (G+H)/\sqrt{2}\,E)^2}{2(GH-EM)/E^2} - \frac{(\mu' + (G-H)/\sqrt{2}\,E)^2}{2(GH-EM)/E^2} = 1,$$

$$\text{if} \quad GH - EM > 0; \tag{13}$$

$$\text{(B)}\quad \text{two asymptotes } \theta = -\frac{H}{E},\ \mu = -\frac{G}{E},$$

$$\text{if}\quad GH - EM = 0; \tag{14}$$

$$\text{(C)}\quad \frac{(\mu' + (G-H)/\sqrt{2}\,E)^2}{2(EM-GH)/E^2} - \frac{(\theta' + (G+H)/\sqrt{2}\,E)^2}{2(EM-GH)/E^2} = 1,$$

$$\text{if}\quad GH - EM < 0. \tag{15}$$

This is an isoaxes hyperbola with

$$\begin{array}{lll} \text{center} & \text{(A)} & \left(-\frac{H}{E}, -\frac{G}{E}\right); \\ & \text{(C)} & \left(-\frac{H}{E}, -\frac{G}{E}\right). \end{array} \tag{16}$$

$$\begin{array}{lll} \text{focuses} & \text{(A)} & \left(-\frac{H}{E} + \frac{\sqrt{2(GH-EM)}}{|E|}, -\frac{G}{E} + \frac{\sqrt{2(GH-EM)}}{|E|}\right), \\ & & \left(-\frac{H}{E} - \frac{\sqrt{2(GH-EM)}}{|E|}, -\frac{G}{E} - \frac{\sqrt{2(GH-EM)}}{|E|}\right); \\ & \text{(C)} & \left(-\frac{H}{E} + \frac{\sqrt{2(EM-GH)}}{|E|}, -\frac{G}{E} - \frac{\sqrt{2(EM-GH)}}{|E|}\right), \\ & & \left(-\frac{H}{E} - \frac{\sqrt{2(EM-GH)}}{|E|}, -\frac{G}{E} + \frac{\sqrt{2(EM-GH)}}{|E|}\right). \end{array} \tag{17}$$

$$\begin{array}{lll} \text{vertexes} & \text{(A)} & \left(-\frac{H}{E} + \frac{\sqrt{GH-EM}}{|E|}, -\frac{G}{E} + \frac{\sqrt{GH-EM}}{|E|}\right), \\ & & \left(-\frac{H}{E} - \frac{\sqrt{GH-EM}}{|E|}, -\frac{G}{E} - \frac{\sqrt{GH-EM}}{|E|}\right); \\ & \text{(C)} & \left(-\frac{H}{E} + \frac{\sqrt{EM-GH}}{|E|}, -\frac{G}{E} - \frac{\sqrt{EM-GH}}{|E|}\right), \\ & & \left(-\frac{H}{E} - \frac{\sqrt{EM-GH}}{|E|}, -\frac{G}{E} + \frac{\sqrt{EM-GH}}{|E|}\right). \end{array} \tag{18}$$

Therefore, the properties of Eqs. (3) and (4) can be presented by the graphs of these hyperbolas (lines or parabolas) and the largest tolerance regions of μ and θ can be found under the restriction of the weighting constraint. Then the critical region can be found correspondingly by incorporating the tolerance level of δ described by relation (9).

2.2. *Sensitivity Analysis*

To facilitate effective control and management, in this section, we analyze the relative levels of sensitivity among parameters and identify the comparatively sensitive parameters.

First, it can be noted that because the interval of δ can be determined independently, one can control the right-hand sides, known as resources, more easily. The consequence is that with other parameters being fixed, one can increase or decrease δ up to the intervals described in (9) but still retain the current optimal solution. However, if there are many types of resources, one should be able to identify more sensitive ones and pay more attention to their variations. In other words, the amount of each resource can be varied within the critical region, but those with relatively smaller tolerance ranges should be under control because they are more sensitive. This relative range of resource b_i can be defined by the tolerance-ratio as $|\Delta b_i/b_i|$. Since $\Delta b = \beta\delta$, thus $|\Delta b_i/b_i| = |(\beta_i/b_i)\,\delta| = |\beta_i/b_i|\,|\delta|$, where δ can be varied in the tolerance interval defined by (9). Therefore for the same range of δ, $|\beta_i/b_i|$ provides a proxy for this comparison of $|\Delta b_i/b_i|$. The smaller the values of $|\beta_i/b_i|$, the more sensitive is the ith resource.

Furthermore, when there is more than one resource with the same level of sensitivity, for optimality, one should look out for those with more unit-contribution to the objective values. Since the shadow price, i.e.,

$$SP(\theta, \mu) \equiv \theta\mu\left[\sum_{i \in I}\left(\sum_{k=1}^{K} \rho^k \alpha_i^k\right) s_{ip}^*\right] + \theta\left[\sum_{i \in I}\left(\sum_{k=1}^{K} \lambda^k \alpha_i^k\right) s_{ip}^*\right] + \mu\left[\sum_{i \in I}\left(\sum_{k=1}^{K} \rho^k c_i^k\right) s_{ip}^*\right] + y_p^*,$$

represents the unit-contribution of the corresponding RHS, we can make use of this relation for comparison. It is noted that the shadow price in form (4) is a second-degree inequality. To compare shadow prices pairwisely, say SP_1 and SP_2, the regional characteristics can be conceived by the difference of SP_1 from SP_2. Because the region with $SP_1 - SP_2 \geqslant 0$ is a second-degree inequality or an equality (hyperbola), the analysis can be carried out analogously to that of finding the critical region of θ and μ. Within the region, it represents the domain where SP_1 is greater than SP_2, so SP_2 is more sensitive; whereas on the boundary, it means that they are equally sensitive. Since the transitivity holds, we can find which resource under what situation is the most sensitive.

Regarding the weighted objective coefficients, the relative tolerance-ratio is defined as $|\Delta C_j/C_j|$. Since $\Delta C_j = \sum_{k=1}^{K} (\lambda^k \alpha_j^k \theta + \rho^k \mu c_j^k + \rho^k \mu \alpha_j^k \theta)$ for $j = 1, \ldots, n$, we have $|\Delta C_j/C_j| = |\sum_{k=1}^{K} (\lambda^k \alpha_j^k \theta + \rho^k \mu c_j^k + \rho^k \mu \alpha_j^k \theta)/C_j|$. In order to find the sensitive term based on these ratios, one can consider two general forms

$$\left|\frac{\Delta C_s}{C_s}\right| \equiv |g_s(\theta, \mu)| = |a_s\theta\mu + b_s\theta + c_s\mu|$$

and

$$\left|\frac{\Delta C_t}{C_t}\right| \equiv |g_t(\theta, \mu)| = |a_t\theta\mu + b_t\theta + c_t\mu|.$$

As in the previous method, if any pair (θ, μ) in the two-dimensional critical region satisfies any of the following conditions, then this set of $\{(\theta, \mu)\}$ confines a region where $|g_s|$ is greater than or equal to $|g_t|$:

$$\begin{aligned} &(1)\quad g_s,\ g_t \geqslant 0 \text{ and } g_s - g_t > 0;\\ &(2)\quad g_s,\ g_t \leqslant 0 \text{ and } g_s - g_t < 0;\\ &(3)\quad g_s \geqslant 0,\ g_t \leqslant 0 \text{ and } g_s + g_t > 0;\\ &(4)\quad g_s \leqslant 0,\ g_t \geqslant 0 \text{ and } g_s + g_t < 0. \end{aligned} \tag{19}$$

Therefore, we say C_t is more sensitive than C_s. Conversely, the remaining area is the region where C_t is more sensitive than C_s. Apparently, the equivalence occurs on the boundary of the region and it means that these two weighted coeffcients are equally important. When two weighted coefficients are equally sensitive, the unit-contribution to the weighted objective function is, in fact, the optimal solution, $x_j^* = b_j^* + \delta \sum_{p=1}^{m} \beta_p s_{jp}^*$. Therefore, with the relative tolerance-ratios being equal, the larger the optimal solution, the more care should be taken. This is, from the second term of the optimal solution, dependent on the critical region of δ defined in (9). Since the transitivity also holds, we can find where and which term of the weighted objective coefficients is the most sensitive.

Furthermore, for the purposes of control and management, the relative sensitivities of costs and weights are our concern. That is, we should look at the inter-relations of θ and μ in the region of the most sensitive term of the weighted objective coefficients obtained above. If it is the jth weighted objective coefficient that is the most sensitive, then we focus on the term of

$$C_j + \Delta C_j = p_j\mu\theta + q_j\theta + r_j\mu + C_j \tag{20}$$

where $p_j = \sum_{k=1}^{K} \rho^k \alpha_j^k$, $q_j = \sum_{k=1}^{K} \lambda^k \alpha_j^k$ and $r_j = \sum_{k=1}^{K} \rho^k c_j^k$. For any fixed value of $C_j + \Delta C_j$, Eq. (20) describes an indifference curve of θ and μ. Thus, we can apply the marginal rate of substitution (MRS) defined by

$$-\left.\frac{\partial\mu}{\partial\theta}\right|_{\text{fix } C_j+\Delta C_j} = \frac{\partial(C_j+\Delta C_j)/\partial\theta}{\partial(C_j+\Delta C_j)/\partial\mu} = \frac{p_j\mu + q_j}{p_j\theta + r_j} \tag{21}$$

to determine the relative tolerance-ratios of θ and μ and its inter-relations. If MRS is negative (positive), then the curve is monotonically increasing (decreasing). When the absolute value of MRS is greater than 1, it means that for one-unit change of θ, μ can be substituted for more than one unit. That is, with the same effort, the variation of μ must be larger than that of θ. In other words, for a fixed value of the weighted objective coefficient, the range we can change in the cost coefficients is less than that in the weights. Therefore, θ is more sensitive and we must take more care with the jth cost coefficients of each objective. The same logic is applied to the cases of $|\text{MRS}| < 1$ of which μ is more sensitive. Thus, a DM's preference structure is required to be more consistent because the variation in giving the weights of importance can not be large. Analogously, if $|\text{MRS}| = 1$, where θ and μ are equally sensitive with a linear relation, then, manipulation of one parameter will proportionally affect the other one.

2.3. *Summary*

From the above, we can summarize the analysis as follows:

(1) For determination of the critical region

(i) the largest tolerance interval of the parameter in the right-hand sides can be independently determined by Eq. (9);

(ii) the largest tolerance region of the parameters in the weights and the cost coefficients dependent on each other and their relations can be defined by the weighting constraint and Eqs. (3) and (4) with their geometric properties of (13)–(18).

Therefore, the largest tolerance region of an MOLP in the form of (2), when global perturbations occur on the weights, the cost coefficients, and the right-hand sides simultaneously, can be determined graphically.

(2) For sensitive analysis, a multi-stage analysis is proposed based on a bi-level criterion.

(i) The sensitive sources of the right-hand sides can be identified with the first level of criterion defined by $|\beta_i/b_i|$; and the second level can be defined by the shadow price in which the cost coefficients and the weights take the effect.

(ii) The relatively sensitive terms of the cost coefficients and the weights can be determined by two stages of analysis

(1st) to identify the sensitive source of the weighted cost coefficients by pairwise comparison with the first level of criterion defined by $|\Delta C_j/C_j|$, and the second level defined by the optimal solution which, in turn, is dependent on δ of the parameter in the RHSs (2);

(2nd) in the most sensitive weighted cost, to determine the relative sensitivities of the costs and weights by their marginal rate of substitution.

Therefore, one can identify which parameter in what substances is the most sensitive and pay attention to their variations for effective control and management.

Let us specify how to apply the analysis above to the following examples so that the critical regions and the sensitive parameters can be identified.

3. Numerical Analysis

EXAMPLE 1.

$$\begin{aligned} \text{Max } z_1 &= -2x_1 - x_2 \\ \text{Max } z_2 &= x_1 + 2x_2 - 2x_3 \\ \text{s.t. } & -x_1 + x_2 + x_3 \leqslant 4 \\ & 2x_1 - x_2 + x_3 \leqslant 2 \\ & x_1 + x_2 + 3x_3 \leqslant 12 \\ & x_1, x_2, x_3 \geqslant 0 \end{aligned} \tag{22}$$

In this example, $c^1 = (-2, -1, 0)$ and $c^2 = (1, 2, -2)$. After performing the weighting method [9, 10], we obtain that the objective function value Z^* increases from 0 to 20 when λ^1 decreases from 1 to 0 and λ^2 increases from 0 to 1. If a decision maker provides the weights of importance with $\lambda^1 = 0.125$ for the first criterion and $\lambda^2 = 0.875$ for the other, then, the problem is changed into

$$\begin{aligned} \text{Max } Z &= 0.625x_1 + 1.625x_2 - 1.75x_3 \\ \text{s.t. } & -x_1 + x_2 + x_3 \leqslant 4 \\ & 2x_1 - x_2 + x_3 \leqslant 2 \\ & x_1 + x_2 + 3x_3 \leqslant 12 \\ & x_1, x_2, x_3 \geqslant 0. \end{aligned} \tag{23}$$

Using the simplex method, we get the final tableau as shown in Table II.

Consider a simultaneous perturbation on the three coefficient vectors with $\beta = (-1, -2, 3)$, $\rho = (1, -1)$, and $\alpha^1 = (-2, 3, 1)$, $\alpha^2 = (1, -2, 1)$.

TABLE II

Final Tableau of Example 1

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
Z	0	0	5.625	0.5	0	1.125	15.5
x_2	0	1	2	0.5	0	0.5	8
x_5	0	0	1	1.5	1	-0.5	2
x_1	1	0	1	-0.5	0	0.5	4

Note. x_1, x_2, x_5 are the optimal basic variables, x_3 is the nonoptimal original variable, and x_4, x_6 are the nonoptimal slack variables. So, $I = \{1, 2\}$, $I' = \{1, 2, 5\}$, $R = \{3\}$, and $P = \{1, 3\}$.

Then, problem (23) becomes

$$\begin{aligned}
\text{Max } Z &= (0.125+\mu)(-2-2\theta, -1+3\theta, \theta)(x_1, x_2, x_3) \\
&\quad + (0.875-\mu)(1+\theta, 2-2\theta, -2+\theta)(x_1, x_2, x_3) \\
&= (0.625 + 0.625\theta - 3\mu - 3\mu\theta)\, x_1 + (1.625 - 1.375\theta - 3\mu + 5\mu\theta)\, x_2 \\
&\quad + (-1.75 + \theta + 2\mu)\, x_3 \\
\text{s.t.} \; & -x_1 + x_2 + x_3 \leqslant 4 - \delta \\
& 2x_1 + x_2 + x_3 \leqslant 2 - 2\delta \\
& x_1 + x_2 + 3x_3 \leqslant 12 + 3\delta \\
& x_1, x_2, x_3 \geqslant 0.
\end{aligned} \tag{24}$$

The weighting constraint can be derived as: $\text{Max}\{\frac{1-0.875}{-1}, \frac{-0.125}{1}\} \leqslant \mu \leqslant \text{Min}\{\frac{-0.875}{-1}, \frac{1-0.125}{1}\}$, that is, $-0.125 \leqslant \mu \leqslant 0.875$.

Now we have the perturbed final Table III.

TABLE III

Perturbed Final Tableau of Example 1

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
Z	0	0	$\frac{45}{8} - \frac{25}{8}\theta - 11\mu + 7\mu\theta$	$\frac{1}{2} - \theta + 4\mu\theta$	0	$\frac{9}{8} - \frac{3}{8}\theta - 3\mu + \mu\theta$	(**)
x_2	0	1	2	0.5		0.5	$8+\delta$
x_5	0	0	1	1.5	1	-0.5	$2-5\delta$
x_1	1	0	1	-0.5	0	0.5	$4+2\delta$

Note. where $(**) = 15.5 - 8.5\theta - 36\mu + 2.875\delta + 28\mu\theta - 9\mu\delta - 0.125\delta\theta - \mu\theta\delta$.

Table III must be feasible and optimal, so

$$5.625 - 3.125\theta - 11\mu + 7\mu\theta \geqslant 0 \tag{25}$$

corresponding to Eq. (3), and

$$0.5 - \theta + 4\mu\theta \geqslant 0 \tag{26}$$

$$1.125 - 0.375\theta - 3\mu + \mu\theta \geqslant 0 \tag{27}$$

corresponding to Eq.(4), and

$$8 + \delta \geqslant 0 \tag{28}$$

$$2 - 5\delta \geqslant 0 \tag{29}$$

$$4 + 2\delta \geqslant 0 \tag{30}$$

corresponding to Eq. (5).

So the largest tolerance interval of δ is $[-2, 2/5]$.

If we apply the result of Eq. (9) directly to find the optimal interval of δ with $I = \{1, 2\}$, $I' = \{1, 2, 5\}$, and $\sum_{p=1}^{3} \beta_p s_{1p}^{*} = 2 > 0$, $\sum_{p=1}^{3} \beta_p s_{2p}^{*} = 1 > 0$, and $\sum_{p=1}^{3} \beta_p s_{5p}^{*} = -5 < 0$, then we have $I^{+} = \{1, 2\}$, $I^{=} = \varnothing$, and $I \ \ = \{5\}$, and

$$\text{Max}\left\{-\frac{8}{1}, -\frac{4}{2}\right\} \leqslant \delta \leqslant \text{Min}\left\{-\frac{2}{-5}\right\}, \qquad \text{i.e.,} \quad \delta \in \left[-2, \frac{2}{5}\right].$$

So, they are consistent.

Now, let us consider the inequalities (25), (26), and (27). When the equalities hold, Eq. (25) is a hyperbola because $E = 7$, $G = -3.125$, $H = -11$, $M = 5.625$, and $GH - EM = -5 < 0$ with asymptotes $\theta = 1.5$, $\mu = 0.47$; center $(1.5, 0.47)$;

$$\text{focuses} \quad \left(1.5 + \frac{\sqrt{10}}{7}, 0.47 - \frac{\sqrt{10}}{7}\right) \quad \text{and} \quad \left(1.5 - \frac{\sqrt{10}}{7}, 0.47 + \frac{\sqrt{10}}{7}\right);$$

$$\text{vertexes} \quad \left(1.5 + \frac{\sqrt{5}}{7}, 0.47 - \frac{\sqrt{5}}{7}\right) \quad \text{and} \quad \left(1.5 - \frac{\sqrt{5}}{7}, 0.47 + \frac{\sqrt{5}}{7}\right).$$

Equation (26) is also a hyperbola because $E = 4$, $G = -1$, $H = 0$, $M = 0$,

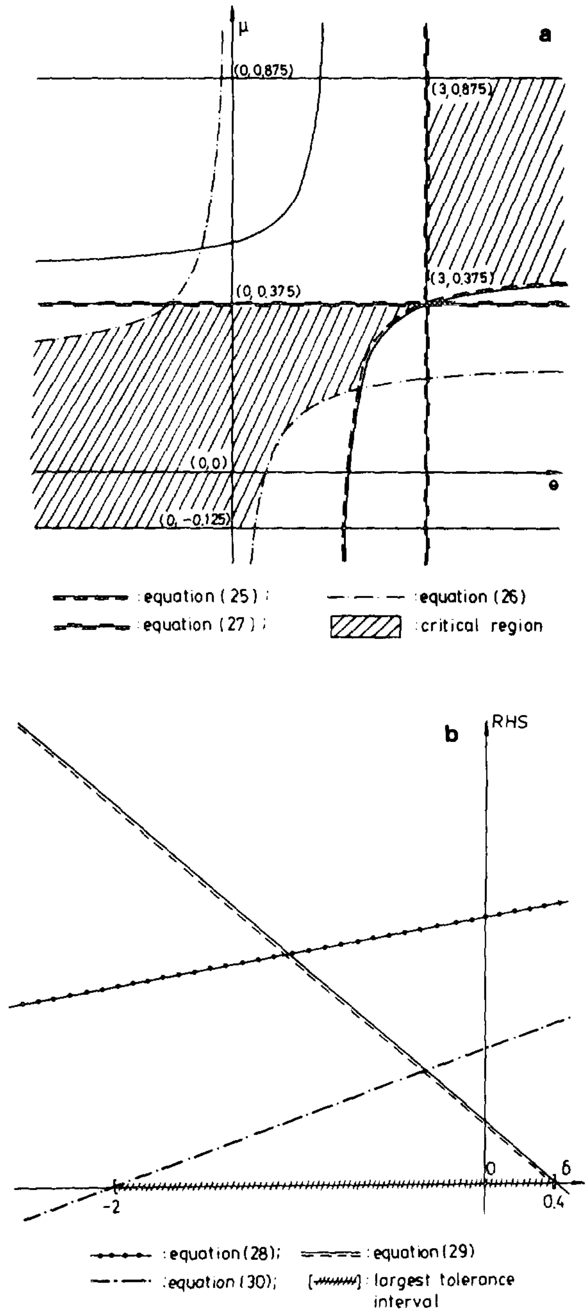


FIG. 1. (a) and (b). The critical region of Example 1.

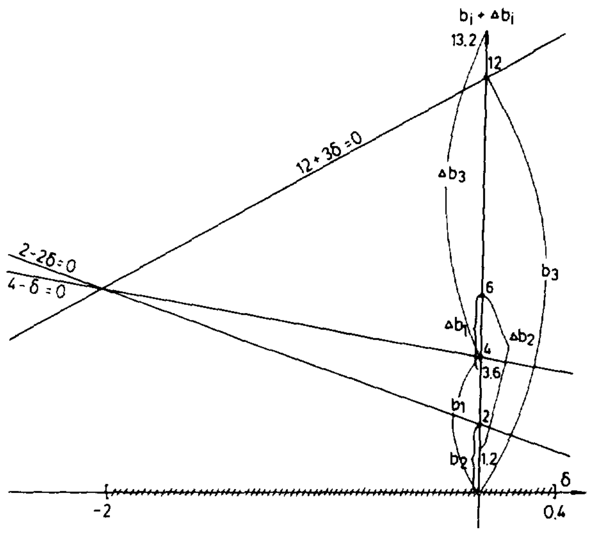


FIG. 2. The relative tolerance-ratios of the RHSs.

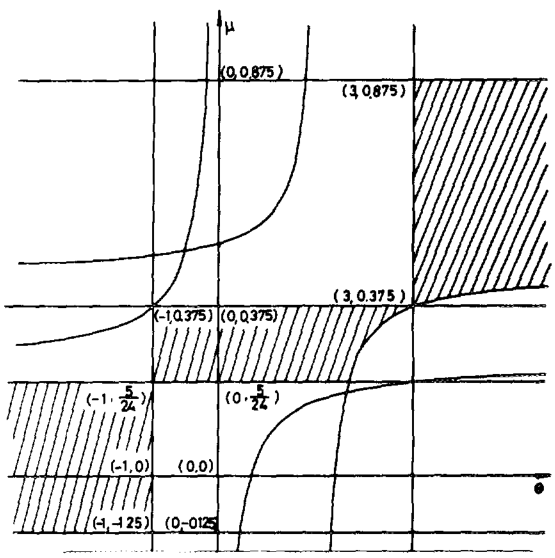


FIG. 3. The shaded area where b_1 has the largest unit-contribution.

and $GH-EM=-2<0$ with asymptotes $\theta=0$ and $\mu=0.25$; center $(0, 0.25)$;

$$\text{focuses} \quad (0.5, -0.25) \quad \text{and} \quad (-0.5, 0.75)$$

$$\text{vertexes} \quad \left(\frac{\sqrt{2}}{4}, 0.25-\frac{\sqrt{2}}{4}\right) \quad \text{and} \quad \left(-\frac{\sqrt{2}}{4}, 0.25+\frac{\sqrt{2}}{4}\right).$$

Equation (27) is degenerate to two asymptotes $\theta=3$, $\mu=0.375$ by $E=1$, $G=-0.375$, $H=-3$, $M=1{,}125$, and $GH-EM=0$.

So, we can plot the graphs and determine the critical region as shown in Fig. 1(a) when δ is mapped into the $\theta-\mu$ plane. The largest tolerance region of θ and μ, which contains $(0, 0)$, is determined by the intersections of hyperbolas defined by Eqs. (25) and (26), and two lines defined by

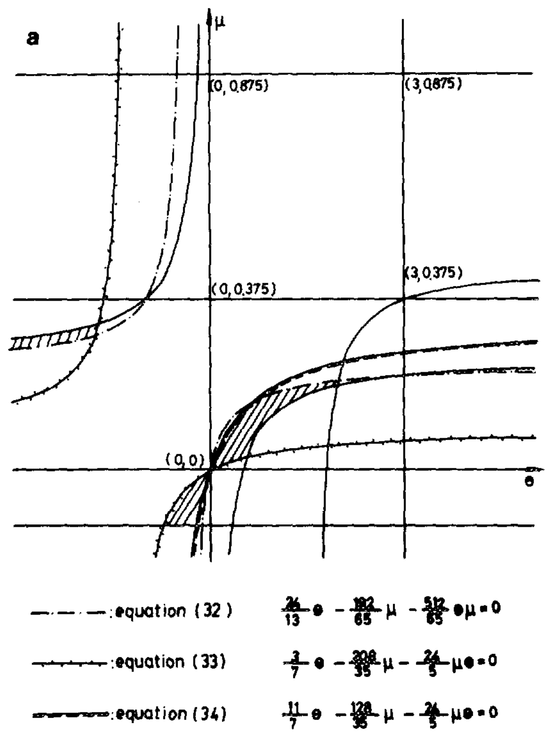
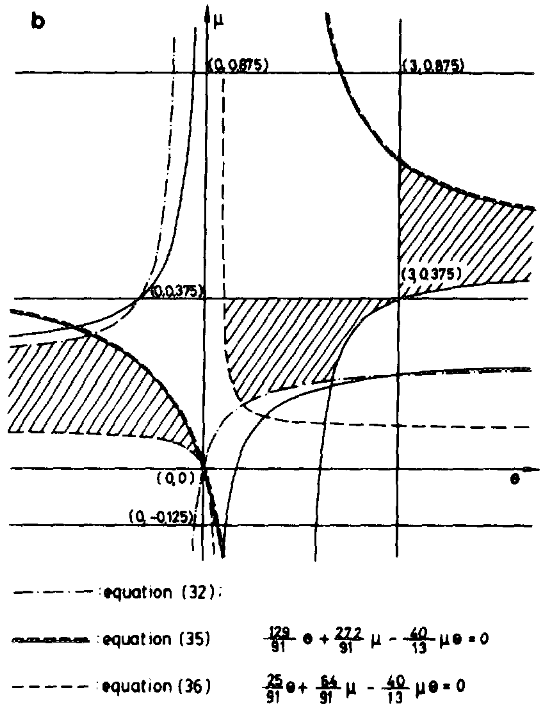


FIG. 4. (a) The shaded area where C_1 has the smallest tolerance-ratio. (b) The shaded area where C_2 has the smallest tolerance-ratio. (c) The shaded area where C_3 has the smallest tolerance-ratio.



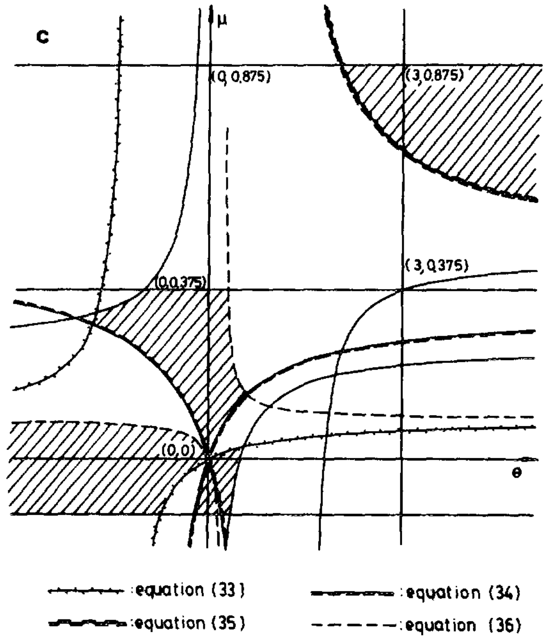


FIG. 4—*Continued*

Eq. (27). Regarding the third dimension of δ, its interval is shown in Fig. 1(b).

Now, in order to identify the sensitive parameter for control purposes, we note that the relative tolerance-ratios of the RHSs are defined by $|\Delta b/b|$ as shown in Fig. 2. Therefore, by $|\beta_1/b_1| = |-1/4| = 0.25$, $|\beta_2/b_2| = |-2/2| = 1$, and $|\beta_3/b_3| = |3/12| = 0.25$, we know that b_1 and b_3 are equally sensitive but more in degree than b_2. In this example, the shadow prices of b_1 and b_3 are $0.5 - \theta + 4\theta\mu$ and $1.125 - 0.375\theta - 3\mu + \theta\mu$, respectively. When (θ, μ) belongs to the shaded area in Fig. 3, the first one is greater than the second one, therefore, we must take more care with b_1; otherwise, b_3 should be paid more attention.

Now, the relative tolerance-ratios of the weighted cost coefficients are

$$\left|\frac{\Delta C_1}{C_1}\right| = \left|\frac{0.625\theta - 3\mu - 3\mu\theta}{0.625}\right| = \left|\theta - \frac{24}{5}\mu - \frac{24}{5}\mu\theta\right|,$$

$$\left|\frac{\Delta C_2}{C_2}\right| = \left|\frac{-1.375\theta - 3\mu + 5\mu\theta}{1.625}\right| = \left|-\frac{11}{13}\theta - \frac{24}{13}\mu + \frac{40}{13}\mu\theta\right|,$$

$$\left|\frac{\Delta C_3}{C_3}\right| = \left|\frac{\theta + 2\mu}{1.75}\right| = \left|\frac{4}{7}\theta + \frac{8}{7}\mu\right|.$$

C_1 behaves the most sensitively in the shaded area of Fig. 4(a); so does C_2 in the shaded area of Fig. 4(b); and C_3 in the shaded area of Fig. 4(c). When C_1 and C_2 are equally sensitive and are more in degree than C_3, we should consider the respective unit-contributions, $x_1^* = 4 + 2\delta$ and $x_2^* = 8 + \delta$. When δ belongs to the largest tolerance interval $[-2, 0.4]$, x_1^* ranges from 0 to 4.8 and x_2^* from 6 to 8.4. Since x_2^* is larger than x_1^*, C_2 is more important and we must take more care with it. Similarly, when C_2 and C_3 are equally sensitive and more in degree than C_1, the unit-contribution of C_3 is 0, thus C_2 is more sensitive. In Fig. 5(a), we use bold lines to show where C_1 and C_3 are equally sensitive and C_1 plays the most important role in the unit-contribution to the objectives. By the same argument, the bold lines of Fig. 5(b) show where C_2 is the most sensitive. Regarding the inter-relation of θ and μ, if we look at the region of the first quadrant where C_2 is the most sensitive, then we are concerned with the behavior of $C_2 + \Delta C_2 = 1.625 - 1.375\theta - 3\mu + 5\mu\theta$. The MRS of $\partial\mu/\partial\theta$ is defined by $(5\mu - 1.375)/(5\theta - 3)$, when $|\text{MRS}| > 1$. That is, if any pair (θ, μ) in the two-dimensional critical region satisfies any of the conditions

$$\begin{aligned} &(1)\quad 5\theta - 5\mu - 1.625 < 0 \text{ or } 5\theta + 5\mu - 4.375 < 0, \text{ if } \theta > 0.6;\\ &(2)\quad 5\theta - 5\mu - 1.625 > 0 \text{ or } 5\theta + 5\mu - 4.375 > 0, \text{ if } \theta < 0.6; \end{aligned} \tag{31}$$

then we can find the corresponding area where θ is more sensitive for positive θ and μ. In other words, we must take more care with the cost

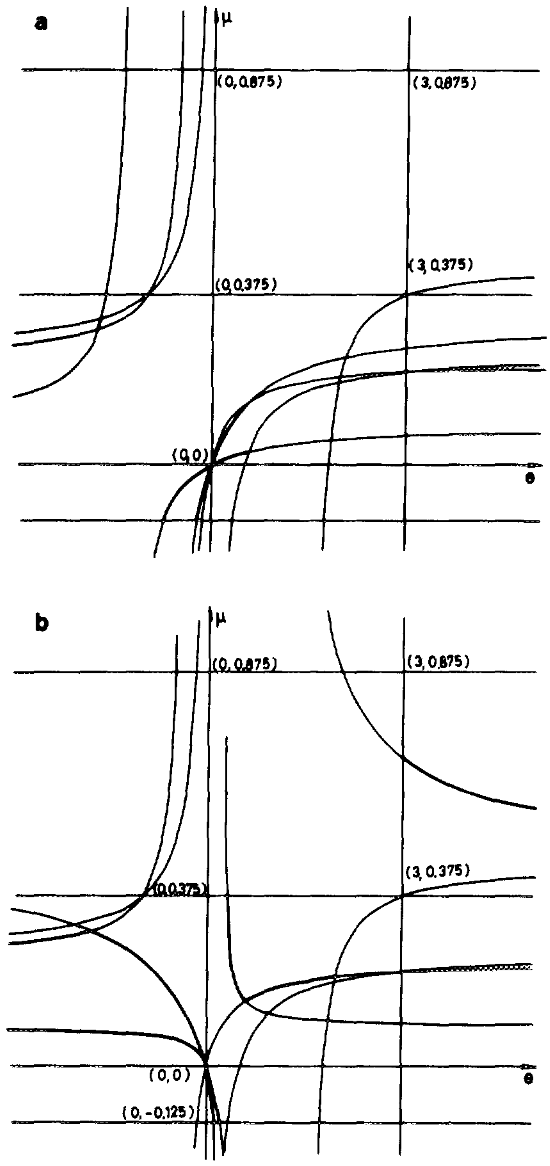


FIG. 5. (a) The boundaries where C_1 has the largest unit-contribution. (b) The boundaries where C_2 has the largest unit-contribution.

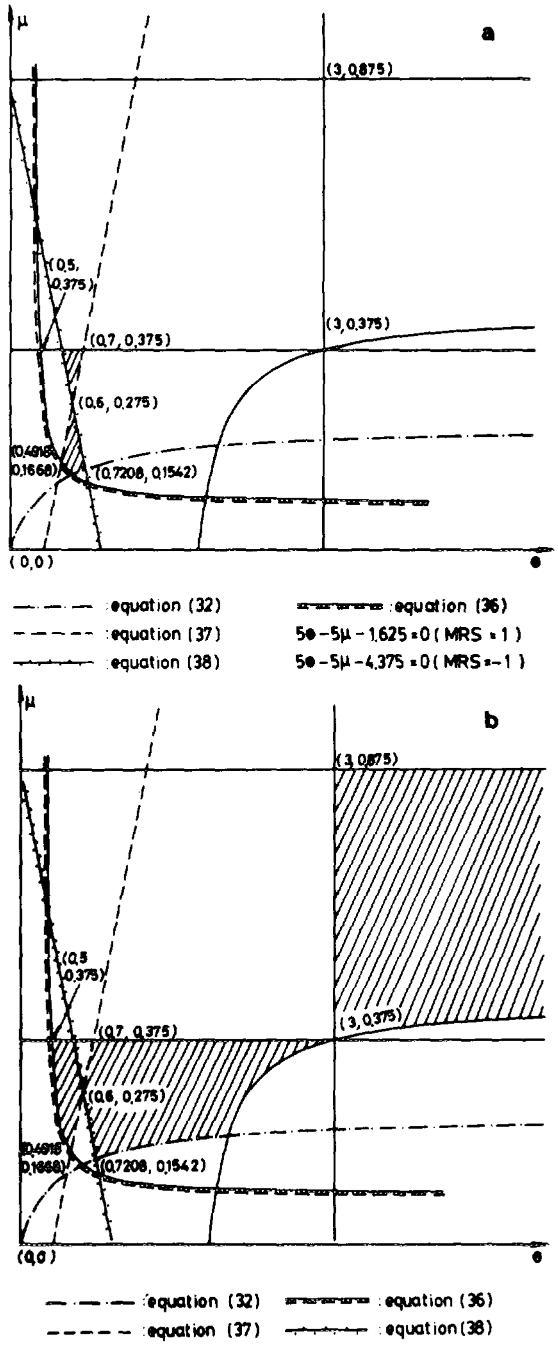


FIG. 6. (a) The region in the first quadrant where θ is more sensitive. (b) The region in the first quadrant where μ is more sensitive.

coefficients. The remaining areas are the regions where μ is the most sensitive. The results are shown in Figs. 6(a) and 6(b). When θ and μ are selected to be in the shaded region in Fig. 6(a), we must take the most care with c_2^1 and c_2^2. In the shaded area of Fig. 6(b), a DM must be careful when giving the weights to each criterion. When θ belongs to (0.4918; 0.77) and (0.5, 0.7208) and $\mu = \theta - 0.325$, $0.875 - \theta$, respectively, both are equally sensitive.

4. Discussions and Conclusions

In this paper, we have performed the inter-parametric tolerance analysis of the combined cost coefficients, the weights, and the RHSs in an MOLP. The analysis is based on the geometric properties so that the critical region, when global perturbation occurs, can be identified and analyzed with the proposed criterion. The relative degrees of sensitivity are significant especially in identifying the sensitive parameters for effective control and management. Therefore, after the maximal tolerance region is defined, a sensitivity analysis is performed to analyze the relative sensitivities of the parameters with the proposed bi-level criterion: (1) the smaller the relative tolerance-ratio, the more sensitive is the degree of a parameter; if two parameters are equally sensitive, for the sake of optimality, (2) the larger the corresponding unit-contribution to the objective function, the more care should be taken with the parameter.

It has been found that when global perturbation occurs, the maximal tolerances of the parameters in the right-hand sides are independent from those of weights and cost coefficients. It is reasonable because changes of the right-hand sides are a kind of parallel displacement and the optimal basis is dependent on the gradients of the objective functions that, in weighted form, are jointly determined by the weights and the cost coefficients. Then, from the first level of the criterion, the relative degrees of sensitivity of the right-hand sides are directly proportional to the given values of β. Regarding the relative degrees of sensitivity of the weights and cost coefficients, because they are inter-related, a regional multi-stage approach is proposed: the first stage is to analyze the joint degrees of sensitivity of the weighted cost coefficients based on the bi-level criterion and they are affected by the parameter in the RHS. After the most sensitive term of the weighted cost coefficients is identified, in the second stage, the relative sensitivies of the costs and the weights are analyzed by applying the concept of a marginal rate of substitution to determine the relative tolerance-ratios. Thus, which parameter in what substances is the most sensitive can be recognized from the graphs.

Potential extensions for further research will be a geometric approach to the sensitivity analysis and inter-parametric tolerance analysis of the four parameters in the cost coefficients, weights, RHSs, and constraint matrix globally and individually. However, it can be foreseen that the high degrees of the relation equations will cause many more complications in analysis.

Besides giving the tolerance interval of each parameter, finding the optimal solution of an MOLP can be a converse and interesting problem to examine because then we will be able to determine the ranges of the solutions for flexible applications. Both are currently under our investigation.

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